Discrete analogue of Fučík spectrum of the Laplacian

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Received 16 May 1997; received in revised form 26 June 1997

Abstract

The discrete analog of the Fučík spectrum for elliptic equations, namely M-matrices, is shown to have properties analogous to the continuum. In particular, the Fučík spectrum of a M-matrix contains a continuous and decreasing curve which is symmetric with respect to the diagonal. © 1999 Elsevier Science B.V. All rights reserved.

Keywords: Fučík spectrum; Discrete Laplacian; M-matrices

1. Introduction

Let $\Omega$ be a bounded domain contained in the plane $\mathbb{R}^2$ and $u$ be defined on $\Omega$ and then define

$$u^+ = \max(u, 0), \quad u^- = \min(u, 0).$$

Then the Fučík spectrum of the Laplacian with a Dirichlet boundary condition is defined as the set $\Sigma$ of those $(\alpha, \beta)$ such that

$$-\Delta u = \alpha u^+ - \beta u^- \quad \text{on} \; \Omega,$$
$$u = 0 \quad \text{on} \; \partial \Omega,$$

has a nontrivial weak solution.

In the discrete case, assume that $x \in \mathbb{R}^n$ and then define

$$x^+ = \max(x, 0), \quad x^- = \min(x, 0).$$

If $A$ is a real $n \times n$ matrix then the discrete analog of the Fučík spectrum for $A$ is the set $\Sigma(n)$ of those $(\alpha, \beta)$ such that

$$Ax = \alpha x^+ - \beta x^- \quad x \in \mathbb{R}^n,$$

has a nontrivial solution. It is well-known that discretizations of (1) are commonly M-matrices, particularly when finite-difference methods are used for the discretization (see [4]). Actually, this
is true, more generally, for elliptic operators in divergence form. So we restrict our attention to
M-matrices.

The discrete analogue $\Sigma(n)$ appears already in Espinoza [3, Proposition 2.3], where the discrete
analogue of semilinear boundary value problems using variational methods are studied. De Figueiredo
and Gossez [1] have obtained a variational characterization of the first nontrivial curve $\Gamma$ contained
in the Fučík Spectrum of a general differential operator in divergence form, with Dirichlet boundary
condition. They have proved that $\Gamma$ is a continuous and strictly decreasing curve, which is symmetric
with respect to the diagonal, unbounded and asymptotic to the lines $\lambda_1 \times [\lambda_1, \infty)$ and $[\lambda_1, \infty) \times \lambda_1$,
where $\lambda_1$ is the first eigenvalue of the differential operator.

2. The discrete analogue

Using the ideas described in [1], we prove that the Fučík spectrum for a $n \times n$ M-matrix $\Sigma(n)$
contains a curve $\Gamma(n)$ with the same properties, except for one, as the curve $\Gamma$ in $\Sigma$. Example 2.8
below shows that the remaining property is not true for the discrete case.

2.1. Some properties of $\Sigma(n)$

M-matrices are defined as in [4]. It is known that M-matrices are real, symmetric, irreducible,
diagonally dominant and positive definite. Hence their eigenvalues are all positive and the first eigen-
value $\lambda_1$ has multiplicity one, with a unique associated norm one eigenvector $\varphi_1$ whose components
are all positive. From this it is easy to see that:

1. The lines $\lambda_1 \times [\lambda_1, \infty)$ and $[\lambda_1, \infty) \times \lambda_1$ are contained in $\Sigma(n)$.
2. Because $(-x)^+ = x^-$ and $(-x)^- = x^+$, $\Sigma(n)$ is symmetric with respect to the diagonal.

Now suppose that $(x, \beta) \in \Sigma(n)$. Then there exists a vector $x \neq 0$ satisfying (2). Multiplying both
sides of (2) by $\varphi_1$, and taking into account that $\varphi_1$ is the eigenvector with eigenvalue $\lambda_1$, gives

$$
\langle (\lambda_1 - \alpha) x^+ - (\lambda_1 - \beta) x^-, \varphi_1 \rangle = 0,
$$

where $\langle \cdot, \cdot \rangle$ is the inner product on $\mathbb{R}^n$ and $\| \cdot \|$ is the norm given by this inner product. If $(\lambda_1 - \alpha) \neq 0$, then

$$
\langle x^+ - r x^-, \varphi_1 \rangle = 0 \quad \text{where} \quad r = \frac{\lambda_1 - \alpha}{\lambda_1 - \beta} x^-,
$$

which says that there exists a orthogonality relationship between the solution $x$ of (2) and the first
eigenvalue $\varphi_1$ of $A$. This relationship motivates the following definitions:

**Definition 1.** For each $r > 0$ define the functions $\psi_1$ and $\psi_2$ on $\mathbb{R}^n$ by

$$
\psi_1(x) = 1 - \|x^+\|^2 - r \|x^-\|^2,
$$

$$
\psi_2(x) = \langle x^+ - r x^-, \varphi_1 \rangle,
$$

$$
and then define the sets $N_r$ and $M_r$ by

$$N_r = \{ x \in \mathbb{R}^n : \psi_1(x) = 0 \},$$
$$M_r = \{ x \in \mathbb{R}^n : \psi_2(x) = 0 \}.$$

**Remark 2.** It is easy to prove the following:

1. $\psi_1(x)$ is a differentiable function and $\psi_2(x)$ is a Lipschitzian and increasing function.
2. $M_r$ is a closed set and $N_r$ is a compact set; hence $M_r \cap N_r$ is a compact set.
3. If $x \in M_r \cap N_r$ then $x^+ \neq 0$ and $x^- \neq 0$. Moreover $\langle x^-, \varphi_1 \rangle \neq 0$ and $\langle x^+, \varphi_1 \rangle \neq 0$.
4. The subspace $\langle \varphi_1 \rangle$ spanned by $\varphi_1$ is disjoint from $M_r \cap N_r$.

**Definition 3.** Define the function $\psi$ by

$$\psi(x) = \langle Ax, x \rangle - \lambda_1 \| x \|^2.$$

The function $\psi$ plays a fundamental role in the study of discrete analogue $\Sigma(n)$ of the Fučík spectrum. Since $\psi(x)$ is a function of class $C^\infty$ on $\mathbb{R}^n$, there exists $\omega \in M_r \cap N_r$, not necessarily unique, such that

$$\psi(\omega) = \min \{ \psi(x) : x \in M_r \cap N_r \}.$$

From Remark 2, $\omega^+ \neq 0$ and $\omega^- \neq 0$, and thus $\omega \notin \langle \varphi_1 \rangle$. Furthermore, as the first eigenvalue $\lambda_1$ of $A$ is positive and strictly less than the other eigenvalues, $\psi(\omega) > 0$.

As $\psi(x)$ and $\psi_1(x)$ are differentiable functions and $\psi_2(x)$ is a Lipschitzian function, by the Lagrange Multiplier Rule (see [2, p. 347]), there exists nonnegative constants $a$, $b$, $c$ and a vector $u$ that belongs to the subgradient (see [2]) of $\psi_2(x)$ at $\omega$ and such that

$$2a(Aw - \lambda_1 \omega) - 2b(\omega^+ - \omega^-) + cu = 0, \quad (3)$$

and $a + b + c = 1$. Multiply both sides of the Eq. (3) by $\varphi_1$ to get $c = 0$, since $\langle u, \varphi_1 \rangle \neq 0$.

If $a = 0$ then $b = 0$, which contradicts the Lagrange Multiplier Rule, so $a \neq 0$. Letting $\theta(r) = b/a$, we have

$$Aw = (\lambda_1 + \theta(r)) \omega^+ - (\lambda_1 + r \theta(r)) \omega^-.$$

Multiplying this equality by $\omega$ gives the explicit form of the function $\theta$:

$$\theta(r) = \psi(\omega) = \min \{ \psi(x) : x \in M_r \cap N_r \}.$$

Thus we have proved the following proposition.

**Proposition 4.** If $M_r$ and $N_r$ are as in Definition 1 then

1. The function $\psi(x)$ given in Definition 3 attains its minimum on $M_r \cap N_r$ at the point $\omega$. Thus there exists a positive function

$$\theta(r) = \min \{ \psi(x) : x \in M_r \cap N_r \} = \psi(\omega)$$

defined for every $r > 0$;
2. \( \omega \) is a solution of the equation

\[ Ax = \alpha x^+ - \beta x^- \]

with

\[ \alpha(r) = \lambda_1 + \theta(r), \quad \beta(r) = \lambda_1 + r\theta(r). \]

Consequently, the curve

\[ \Gamma(n) = \{(\alpha(r), \beta(r)): r > 0\} \]

is contained in the discrete analogue of the Fučík spectrum: \( \Sigma(n) \).

Now we will prove the continuity of \( \theta(r) \) for \( r > 0 \). Recall that \( \alpha(r) \) and \( \beta(r) \) are continuous functions for every \( r > 0 \). So let \( r_m \) be a sequence of real positive numbers that converge to \( r > 0 \). Also let \( \omega_m \in M_{r_m} \cap N_{r_m} \) and \( \omega \in M_r \cap N_r \) be points where \( \theta(r_m) = \psi(\omega_m) \) and \( \theta(r) = \psi(\omega) \). Since \( \omega_m \) is bounded, there exists a subsequence (denoted in the same way) and \( v \in \mathbb{R}^n \) such that \( \omega_m \to v \). Because of the continuity of \( \psi(x) \), \( \psi_2(x) \) and \( \psi(x) \) it follows that \( v \in M_r \cap N_r \) and \( \theta(r_m) = \psi(\omega_m) \to \psi(v) \).

On the other hand, it is quite easy to prove that there exists \( \alpha_m \) and \( \beta_m \), positive real numbers, such that

\[ z_m = \alpha_m \omega^+ - \beta_m \omega^+ \in M_{r_m} \cap N_{r_m} \cdot \]

Hence \( \psi(\omega_m) \leq \psi(z_m) \). Moreover, since \( (\alpha_m, \beta_m) \to (1, 1) \), and \( \psi(v) \leq \psi(\omega) \), the continuity of \( \theta(r) \) then follows. It is easy to check that \( x \in M_r \cap N_r \) if only if

\[ (-\sqrt{r} \cdot x) \in M_{1/r} \cap N_{1/r}, \]

from which it follows \( r \psi(x) = \psi(-\sqrt{r} \cdot x) \) and consequently \( r^{\theta(r)} = r(1/r) \).

Finally, we conclude this section by showing that \( \Gamma(n) \) is a decreasing curve. The functions \( g \) and \( h \) are defined as

**Definition 5.**

\[ g(\varepsilon) = \| (x + \varepsilon \varphi_1^+) + (x + \varepsilon \varphi_1^-) \|^2, \]

\[ h(\varepsilon) = \| (x + \varepsilon \varphi_1^+)^2 + (r + \varepsilon + x + \varepsilon \varphi_1^-) \|^2. \]

It is easy to check that \( g(\varepsilon) > 1 \) for all \( \varepsilon > 0 \) and thus \( h(\varepsilon) > 1 \). There exists \( \varepsilon > 0 \) such that

\[ z = \omega + \varepsilon \varphi_1 \in M_{r+1} \cap N_{r+1}, \]

and then also

\[ \frac{z}{\sqrt{h(\varepsilon)}} \in M_{r+1} \cap N_{r+1}. \]

Hence

\[ \theta(r) \leq \psi \left( \frac{z}{\sqrt{h(\varepsilon)}} \right) = \frac{1}{h(\varepsilon)} \psi(z) = \frac{1}{h(\varepsilon)} \psi(\omega) < \psi(\omega) = \theta(r). \]
Thus we have proved the proposition:

**Proposition 6.** Let $\alpha(r)$, $\beta(r)$ and $\Gamma(n)$ be as in Proposition 2.6. Then for every $r > 0$
(a) $\alpha(r) = \beta(1/r)$;
(b) $\alpha(r)$ and $\beta(r)$ are continuous functions;
(c) $\alpha(r)$ is a decreasing function.

**Example 7.** Consider the matrix

\[
A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.
\]

In this case $\lambda_1 = 1$ and

\[
\Gamma(n) = \left\{ (2 + \frac{1}{r^2} + r) : r > 0 \right\}.
\]

This is a curve which is not asymptotic to the lines $\lambda_1 \times [\lambda_1, \infty)$ and $[\lambda_1, \infty) \times \lambda_1$.

Consequently, the asymptotic property in the continuum cannot hold in the discrete case.

**References**


